

REPRESENTATIONS OF 0-YOKONUMA-HECKE ALGEBRAS

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ABSTRACT. We give two different approaches to classifying the simple modules of 0-Yokonuma-Hecke algebras $Y_{r,n}(0)$ over an algebraically closed field of characteristic p such that p does not divide r . Using the isomorphism between the 0-Yokonuma-Hecke algebra and 0-Ariki-Koike-Shoji algebra, we in fact give another way to obtain the simple modules of the latter, which was previously studied by Hivert, Novelli and Thibon (Adv. Math. **205** (2006) 504-548). In the appendix, we give the classification of simple modules of the nil Yokonuma-Hecke algebra.

1. INTRODUCTION

1.1. The Iwahori-Hecke algebra $\mathcal{H}_q(W)$ can be regarded as a q -deformation of the group algebra of a finite Coxeter group W , which arises in the study of groups with (B, N) -pairs and plays an important role in the study of representations of finite groups of Lie type.

When $q = 0$, we get the so-called 0-Hecke algebra $\mathcal{H}_0(W)$, whose representation theory is very different from that of $\mathcal{H}_q(W)$ with a non-zero parameter q . In [No], Norton classified the irreducible modules and indecomposable projective modules of $\mathcal{H}_0(W)$. In [Ca], Carter gave the decomposition numbers of it in type A . $\mathcal{H}_0(W)$ has been studied extensively by various people; see [DHT, DY, Fa, He, HNT, Hu, KT, YL] and so on.

1.2. Yokonuma-Hecke algebras were introduced by Yokonuma [Yo] as a centralizer algebra associated to the permutation representation of a finite Chevalley group G with respect to a maximal unipotent subgroup of G . By the presentation given by Juyumaya and Kannan [Ju1, Ju2, JuK], the Yokonuma-Hecke algebra $Y_{r,n}(q)$ (of type A) can be regarded as a deformation of the group algebra of the complex reflection group $G(r, 1, n)$, which is isomorphic to the wreath product $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$, where \mathfrak{S}_n is the symmetric group. Yokonuma-Hecke algebras have been studied in [ChPA, C, JaPA, Lu] and so on.

The modified Ariki-Koike algebra [SS], as a way of approximating the usual Ariki-Koike algebra, was first defined by Shoji [S] in order to give a Frobenius type formula for the characters of Ariki-Koike algebras. In [HNT], they studied the representation theory of the modified Ariki-Koike algebra when the parameter $q = 0$, called 0-Ariki-Koike-Shoji algebra; they classified its simple modules and projective modules, and described its Cartan invariants and decomposition matrices. Later on, Espinoza and Ryom-Hansen [ER] proved that the Yokonuma-Hecke algebra is isomorphic to the modified Ariki-Koike algebra when the parameter q is invertible.

In this note, we shall consider the particular case $Y_{r,n}(0)$ when $q = 0$, which we call 0-Yokonuma-Hecke algebras. In fact, we can easily see that we also have an isomorphism between the two algebras mentioned above when $q = 0$, that is, the 0-Yokonuma-Hecke

algebra is isomorphic to the 0-Ariki-Koike-Shoji algebra with a particular choice of the parameters. In particular, we give two different approaches to classifying the simple modules of 0-Yokonuma-Hecke algebras $Y_{r,n}(0)$ over an algebraically closed field of characteristic p such that p does not divide r . Using the isomorphism between 0-Yokonuma-Hecke algebras and 0-Ariki-Koike-Shoji algebras, we in fact give another way to obtain the simple modules of the latter, which were previously studied in [HNT].

2. PRELIMINARIES

2.1. 0-Yokonuma-Hecke algebras. Let $r, n \in \mathbb{N}$, $r, n \geq 1$, and let $\zeta = e^{2\pi i/r}$. Let \mathbb{K} be an algebraically closed field of characteristic $p \geq 0$ such that p does not divide r , which contains ζ and an arbitrary element q .

The Yokonuma-Hecke algebra $Y_{r,n}(q)$ is a \mathbb{K} -associative algebra generated by the elements $t_1, \dots, t_n, g_1, \dots, g_{n-1}$ satisfying the following relations:

$$\begin{aligned} t_i^r &= 1 & \text{for all } i = 1, \dots, n; \\ t_i t_j &= t_j t_i & \text{for all } i, j = 1, \dots, n; \\ g_i t_j &= t_{s_i(j)} g_i & \text{for all } i = 1, \dots, n-1 \text{ and } j = 1, \dots, n; \\ g_i g_j &= g_j g_i & \text{for all } i, j = 1, \dots, n-1 \text{ such that } |i-j| \geq 2; \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} & \text{for all } i = 1, \dots, n-2; \\ g_i^2 &= q + (q-1)e_i g_i & \text{for all } i = 1, \dots, n-1, \end{aligned} \tag{2.1}$$

where s_i is the transposition $(i, i+1)$, and for each $1 \leq i \leq n-1$,

$$e_i := \frac{1}{r} \sum_{s=0}^{r-1} t_i^s t_{i+1}^{-s}.$$

Note that the elements e_i are idempotents in $Y_{r,n}(q)$. For each $w \in \mathfrak{S}_n$, let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression of w . By Matsumoto's lemma, the element $g_w := g_{i_1} g_{i_2} \cdots g_{i_r}$ does not depend on the choice of the reduced expression of w , that is, it is well-defined. Moreover, the following elements

$$\{t_1^{a_1} \cdots t_n^{a_n} g_w \mid 1 \leq a_1, \dots, a_n \leq r \text{ and } w \in \mathfrak{S}_n\} \tag{2.2}$$

form a \mathbb{K} -basis of $Y_{r,n}(q)$.

Let $i, k \in \{1, 2, \dots, n\}$ and set

$$e_{i,k} := \frac{1}{r} \sum_{s=0}^{r-1} t_i^s t_k^{-s}. \tag{2.3}$$

Note that $e_{i,k}^2 = e_{i,k} = e_{k,i}$, and that $e_{i,i+1} = e_i$. It can be easily checked that

$$\begin{aligned} t_i e_{j,k} &= e_{j,k} t_i & \text{for all } i, j, k = 1, \dots, n, \\ e_{i,j} e_{k,l} &= e_{k,l} e_{i,j} & \text{for all } i, j, k, l = 1, \dots, n, \\ e_i e_{k,l} &= e_{s_i(k), s_i(l)} e_i & \text{for all } i = 1, \dots, n-1 \text{ and } k, l = 1, \dots, n, \\ e_{j,k} g_i &= g_i e_{s_i(j), s_i(k)} & \text{for all } i = 1, \dots, n-1 \text{ and } j, k = 1, \dots, n. \end{aligned} \tag{2.4}$$

In particular, we have $e_i g_i = g_i e_i$ for all $i = 1, 2, \dots, n-1$.

Let \mathcal{T} be the commutative subalgebra of $Y_{r,n}(q)$ generated by t_1, \dots, t_n which is isomorphic to the group algebra of $(\mathbb{Z}/r\mathbb{Z})^n$ over \mathbb{K} . A character χ of \mathcal{T} over \mathbb{K} is determined

by the choice of $\chi(t_j) \in \{\zeta_1, \dots, \zeta_r\}$ for $1 \leq j \leq n$, where $\zeta_i = \zeta^i$ for $1 \leq i \leq r$. Let $\text{Irr}(\mathcal{T})$ denote the set of characters of \mathcal{T} over \mathbb{K} . The symmetric group \mathfrak{S}_n acts by permutations on \mathcal{T} and induces an action on $\text{Irr}(\mathcal{T})$ given by $w(\chi)(t_i) = \chi(t_{w^{-1}(i)})$ for $1 \leq i \leq n$, $w \in \mathfrak{S}_n$ and $\chi \in \text{Irr}(\mathcal{T})$.

For each $\chi \in \text{Irr}(\mathcal{T})$, let E_χ be the primitive idempotent of \mathcal{T} associated to χ , that is, $\chi'(E_\chi) = 0$ if $\chi' \neq \chi$ and $\chi(E_\chi) = 1$. Then E_χ can be explicitly written in terms of the generators as follows:

$$E_\chi = \prod_{1 \leq i \leq n} \left(\frac{1}{r} \sum_{0 \leq s \leq r-1} \chi(t_i)^s t_i^{-s} \right). \quad (2.5)$$

By definition, we have, for each $1 \leq i \leq n$, $w \in \mathfrak{S}_n$ and $\chi \in \text{Irr}(\mathcal{T})$,

$$t_i E_\chi = E_\chi t_i = \chi(t_i) E_\chi \quad \text{and} \quad g_w E_\chi = E_{w(\chi)} g_w. \quad (2.6)$$

Thus, we immediately get that

$$t_i = \sum_{\chi \in \text{Irr}(\mathcal{T})} \chi(t_i) E_\chi \quad \text{for each } 1 \leq i \leq n. \quad (2.7)$$

The following presentation of $Y_{r,n}(q)$ is proved in [ER, Proposition 2] when q is invertible, which in fact is a particular case of unipotent Hecke algebras considered in [Lu, §31.2].

Proposition 2.1. (See [ER, Proposition 2].) *$Y_{r,n}(q)$ has a second presentation, which is generated by g_1, \dots, g_{n-1} and E_χ ($\chi \in \text{Irr}(\mathcal{T})$) with relations:*

$$\begin{aligned} \sum_{\chi \in \text{Irr}(\mathcal{T})} E_\chi &= 1; \\ E_{\chi'} E_\chi &= \delta_{\chi', \chi} E_\chi; \\ g_i E_\chi &= E_{s_i(\chi)} g_i && \text{for all } 1 \leq i \leq n-1; \\ g_i g_j &= g_j g_i && \text{for all } 1 \leq i, j \leq n-1 \text{ such that } |i-j| > 1; \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} && \text{for all } 1 \leq i \leq n-2; \\ g_i^2 &= q + (q-1) \sum_{\substack{\chi \in \text{Irr}(\mathcal{T}) \\ s_i(\chi) = \chi}} E_\chi g_i && \text{for all } 1 \leq i \leq n-1. \end{aligned}$$

By (2.2), we can easily get that the following elements

$$\{E_\chi g_w \mid \chi \in \text{Irr}(\mathcal{T}) \text{ and } w \in \mathfrak{S}_n\} \quad (2.8)$$

form a \mathbb{K} -basis of $Y_{r,n}(q)$.

The following presentation of $Y_{r,n}(q)$ is proved in [ER, Theorem 7] when q is invertible, which claims that $Y_{r,n}(q)$ is isomorphic to the modified Ariki-Koike algebra defined in [S, Section 3.6] with a particular choice of the parameters u_i , that is, $u_i = \zeta^i$ for $1 \leq i \leq r$.

Proposition 2.2. (See [ER, Theorem 7].) $Y_{r,n}(q)$ has a third presentation, which is generated by h_1, \dots, h_{n-1} and w_1, \dots, w_n with relations:

$$\begin{aligned}
w_i^r &= 1 && \text{for all } 1 \leq i \leq n; \\
w_i w_j &= w_j w_i && \text{for all } 1 \leq i, j \leq n; \\
h_i h_j &= h_j h_i && \text{for all } 1 \leq i, j \leq n-1 \text{ such that } |i-j| > 1; \\
h_i h_{i+1} h_i &= h_{i+1} h_i h_{i+1} && \text{for all } 1 \leq i \leq n-2; \\
h_i w_i &= w_{i+1} h_i - \Delta^{-2} \sum_{c_1 < c_2} (\zeta^{c_2} - \zeta^{c_1})(q-1) F_{c_1}(w_i) F_{c_2}(w_{i+1}) && \text{for all } 1 \leq i \leq n-1; \\
h_i(w_i + w_{i+1}) &= (w_i + w_{i+1}) h_i && \text{for all } 1 \leq i \leq n-1; \\
h_i w_l &= w_l h_i && \text{for all } l \neq i, i+1 \text{ and } 1 \leq i \leq n-1; \\
h_i^2 &= q + (q-1) h_i && \text{for all } 1 \leq i \leq n-1.
\end{aligned}$$

For each $w \in \mathfrak{S}_n$, let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression of w . By Matsumoto's lemma again, the element $h_w := h_{i_1} h_{i_2} \cdots h_{i_r}$ is well-defined. Moreover, It is known by [S] that the following set

$$\{w_1^{k_1} \cdots w_n^{k_n} h_w \mid 1 \leq k_1, \dots, k_n \leq r \text{ and } w \in \mathfrak{S}_n\} \quad (2.9)$$

gives rise to a \mathbb{K} -basis of $Y_{r,n}(q)$.

For each $\chi \in \text{Irr}(\mathcal{T})$, since χ is completely determined by the values $(\chi(t_1), \dots, \chi(t_n))$ with each $\chi(t_k) = \zeta^{c_k}$ for $1 \leq k \leq n$ and $1 \leq c_k \leq r$. Thus, we can identify each χ with $c = (c_1, \dots, c_n) \in C^n$ such that $\chi(t_k) = \zeta^{c_k}$ for $1 \leq k \leq n$, where $C = \{1, \dots, r\}$.

For each $1 \leq k \leq r$, we denote by $P_k(X)$ the Lagrange polynomial

$$P_k(X) = \prod_{1 \leq l \leq r; l \neq k} \frac{X - \zeta^l}{\zeta^k - \zeta^l}.$$

The following lemma can be easily proved by definition.

Lemma 2.3. *If we identify $\chi \in \text{Irr}(\mathcal{T})$ with $c \in C^n$. Then we have*

$$E_\chi = P_{c_1}(t_1) \cdots P_{c_n}(t_n).$$

For each $c = (c_1, \dots, c_n) \in C^n$, we define

$$L_c := P_{c_1}(w_1) \cdots P_{c_n}(w_n).$$

The following proposition can be easily obtained by using [HNT, Lemma 4.1] and Proposition 2.2.

Proposition 2.4. $Y_{r,n}(q)$ has a forth presentation, which is generated by h_1, \dots, h_{n-1} and L_c ($c \in C^n$) with relations:

$$\begin{aligned} \sum_{c \in C^n} L_c &= 1; \\ L_{c'} L_c &= \delta_{c',c} L_c; \\ h_i h_j &= h_j h_i && \text{for all } 1 \leq i, j \leq n-1 \text{ such that } |i-j| > 1; \\ h_i h_{i+1} h_i &= h_{i+1} h_i h_{i+1} && \text{for all } 1 \leq i \leq n-2; \\ h_i^2 &= q + (q-1)h_i && \text{for all } 1 \leq i \leq n-1; \\ h_i L_c &= L_{s_i(c)} h_i - (q-1) \begin{cases} -L_c & \text{if } c_i < c_{i+1}, \\ 0 & \text{if } c_i = c_{i+1}, \\ L_{s_i(c)} & \text{if } c_i > c_{i+1}, \end{cases} \end{aligned}$$

where s_i acts on c by exchanging c_i and c_{i+1} .

By [HNT, Proposition 4.2], the following set

$$\{L_c h_w \mid c \in C^n \text{ and } w \in \mathfrak{S}_n\} \quad (2.10)$$

also gives rise to a \mathbb{K} -basis of $Y_{r,n}(q)$.

In this paper, we consider the particular case $Y_{r,n}(0)$ when $q = 0$, which we call 0-Yokonuma-Hecke algebras.

2.2. Classification of simple modules. For each $c = (c_1, \dots, c_n) \in C^n$, we define an associative set $I_c = ((I_c)_1, \dots, (I_c)_k)$, where $(I_c)_j = (c_{i_{j-1}+1}, \dots, c_{i_j})$ is such that $c_{i_{j-1}+1} = \dots = c_{i_j}$ and $c_{i_j} \neq c_{i_{j+1}}$ for $1 \leq j \leq k$, and where $i_0 = 0$ and $i_k = n$. We also define $|(I_c)_j| = i_j - i_{j-1}$ for $1 \leq j \leq k$.

Theorem 2.5. *All the simple $Y_{r,n}(0)$ -modules are of dimension 1. Moreover, they are indexed by the following set:*

$$\{(c, J) = ((c_1, \dots, c_n), (J_1, \dots, J_k)) \mid c \in C^n \text{ and } J_j \text{ is a composition of } |(I_c)_j| \text{ for } 1 \leq j \leq k\}.$$

Proof. By definition, we can easily check that the following equalities hold.

- (1) $(g_i g_{i+1} - g_{i+1} g_i)^3 = 0$;
- (2) $(g_i t_i - t_i g_i)^2 = (g_i t_{i+1} - t_{i+1} g_i)^2 = 0$.

Using (2.4), (1) and (2), it involves making a lengthy but routine calculation to get that the commutators $[g_i, g_{i+1}]$ and $[g_i, t_i]$ are strongly nilpotent elements. Let J denote the two-sided ideal generated by all the commutators $[g_i, g_j]$ and $[g_i, t_j]$. Thus, we have $J \subseteq \text{rad}(Y_{r,n}(0))$. Let $\overline{Y}_{r,n}(0) = Y_{r,n}(0)/J$. Then $\overline{Y}_{r,n}(0)$ is a commutative algebra generated by $\overline{t}_1, \dots, \overline{t}_n$ and $\overline{g}_1, \dots, \overline{g}_{n-1}$ with relations:

$$\overline{t}_i^r = 1, \quad \overline{t}_i \overline{t}_j = \overline{t}_j \overline{t}_i, \quad \overline{g}_i \overline{t}_j = \overline{t}_j \overline{g}_i, \quad \overline{g}_i \overline{t}_i = \overline{t}_{i+1} \overline{g}_i, \quad \overline{g}_i \overline{g}_j = \overline{g}_j \overline{g}_i \quad \text{and} \quad \overline{g}_i^2 = -\overline{g}_i.$$

It is easy to see that $\overline{Y}_{r,n}(0)$ has no nilpotent elements. So $\overline{Y}_{r,n}(0)$ is semisimple and $J \supseteq \text{rad}(Y_{r,n}(0))$. Therefore, $J = \text{rad}(Y_{r,n}(0))$, and $\overline{Y}_{r,n}(0) = Y_{r,n}(0)/\text{rad}(Y_{r,n}(0))$.

Since $\overline{Y}_{r,n}(0)$ is commutative, all the simple $Y_{r,n}(0)$ -modules are of dimension 1, and moreover, they are indexed by the set (c, J) . In fact, for each (c, J) such that $c \in$

C^n and J_j is a composition of $|(I_c)_j|$ for $1 \leq j \leq k$, let $D(J_i)$ be the associated subset of $[1, |(I_c)_j| - 1]$. Then the associated irreducible representation $\varphi_{(c,J)}$ of $Y_{r,n}(0)$ is defined by

$$\varphi_{(c,J)}(t_i) = \zeta^{c_i} \quad \text{and} \quad \varphi_{(c,J)}(g_k) = \begin{cases} -1 & \text{if } k \in \sum_{1 \leq l \leq j-1} |(I_c)_l| + D(J_j) \text{ for some } j, \\ 0 & \text{otherwise.} \end{cases}$$

□

2.3. $Y_{r,n}(0)$ is Frobenius. Recall that an algebra A over \mathbb{K} is called Frobenius if there is a linear map $\tau : A \rightarrow \mathbb{K}$ whose kernel contains no non-zero left or right ideal of A . The following proposition claims that $Y_{r,n}(0)$ is Frobenius, and is in particular self-injective.

Proposition 2.6. *$Y_{r,n}(0)$ is a Frobenius algebra.*

Proof. We define a linear map $\tau : Y_{r,n}(0) \rightarrow \mathbb{K}$ by

$$\tau(t_1^{a_1} \cdots t_n^{a_n} g_w) = \begin{cases} 1 & \text{if } w = w_0, \\ 0 & \text{otherwise.} \end{cases} \quad (2.11)$$

where w_0 is the longest element of \mathfrak{S}_n . We then claim that for any $0 \neq h \in Y_{r,n}(0)$, there exist elements j and k such that $\tau(jh)$ and $\tau(hk)$ are non-zero. By (2.2), we can write h as a linear combination of some elements $g_y t_1^{b_1} \cdots t_n^{b_n}$, and assume that w is an element of maximal length such that $g_w t_1^{c_1} \cdots t_n^{c_n}$ occurs with non-zero coefficient. Let $j = g_{w_0 w^{-1}}$ and $k = g_{w^{-1} w_0}$.

Since we have

$$g_i g_w = \begin{cases} g_{s_i w} & \text{if } \ell(s_i w) > \ell(w), \\ -e_i g_w & \text{if } \ell(s_i w) < \ell(w), \end{cases}$$

we notice that for any $x, y \in \mathfrak{S}_n$, $g_x g_y$ is of the form $t g_z$ such that $\ell(z) \leq \ell(x) + \ell(y)$, and $\ell(z) = \ell(x) + \ell(y)$ if and only if $\ell(xy) = \ell(x) + \ell(y)$, in which case $z = xy$ and $t = 1$. Thus, we get that $j \cdot g_w t_1^{c_1} \cdots t_n^{c_n} = g_{w_0} t_1^{c_1} \cdots t_n^{c_n}$ and $g_w t_1^{c_1} \cdots t_n^{c_n} \cdot k = g_{w_0} t_1^{c'_1} \cdots t_n^{c'_n}$, while $\tau(j \cdot g_x t_1^{b_1} \cdots t_n^{b_n}) = 0 = \tau(g_x t_1^{b_1} \cdots t_n^{b_n} \cdot k)$ for any $x \neq w$ with $\ell(x) \leq \ell(w)$. □

There is an involution ϕ on $Y_{r,n}(0)$ defined by $\phi(g_i) = g_{n-i}$ for $1 \leq i \leq n-1$ and $\phi(t_j) = t_{n+1-j}$ for $1 \leq j \leq n$. Then we have the following result.

Proposition 2.7. *Let $\tau : Y_{r,n}(0) \rightarrow \mathbb{K}$ be defined as in (2.11). Then for all a and b in $Y_{r,n}(0)$, we have $\tau(ab) = \tau(\phi(b)a)$.*

2.4. $Y_{r,n}(0)$ is standardly based. Recall that the notion of a standardly based algebra was introduced in [DR, Definition 1.2.1] and a complete classification of a finite dimensional standardly based algebra over a field is also provided in [DR, Theorem 2.4.1]. The proof of the following theorem is inspired by that of [YL, Theorem 5.1].

Theorem 2.8. *$Y_{r,n}(0)$ is a standardly based algebra with a standard basis $\{E_\chi g_w \mid \chi \in \text{Irr}(\mathcal{T}) \text{ and } w \in \mathfrak{S}_n\}$. Moreover, the simple modules of $Y_{r,n}(0)$ over \mathbb{K} are exactly those which are given in Theorem 2.5.*

Proof. Since we have

$$g_i g_w = \begin{cases} g_{s_i w} & \text{if } \ell(s_i w) > \ell(w), \\ -\sum_{\chi: s_i(\chi) = \chi} E_\chi g_w & \text{if } \ell(s_i w) < \ell(w), \end{cases}$$

we get that

$$E_{\chi'} g_i \cdot E_\chi g_w = E_{\chi'} E_{s_i(\chi)} g_i g_w = \begin{cases} 0 & \text{if } s_i(\chi) \neq \chi', \\ E_{\chi'} g_{s_i w} & \text{if } s_i(\chi) = \chi' \text{ and } \ell(s_i w) > \ell(w), \\ -E_{\chi'} g_w & \text{if } s_i(\chi) = \chi' = \chi \text{ and } \ell(s_i w) < \ell(w), \\ 0 & \text{if } s_i(\chi) = \chi' \neq \chi \text{ and } \ell(s_i w) < \ell(w). \end{cases} \quad (2.12)$$

Set $\Lambda := \{(\chi, w) \mid \chi \in \text{Irr}(\mathcal{T}) \text{ and } w \in \mathfrak{S}_n\}$. We fix a total ordering on the elements of \mathfrak{S}_n such that $w_i > w_j$ whenever $\ell(w_i) > \ell(w_j)$ and extend it to Λ . Assume that $I(\chi, w)$ and $J(\chi, w)$ consist of only one element $E_\chi g_w$ for each $(\chi, w) \in \Lambda$. By (2.12), we see that the basis $\{E_\chi g_w\}$ is a standard basis.

In order to classify the simple modules of $Y_{r,n}(0)$ using [DR, Theorem 2.4.1], we must choose those (χ, w) such that $\beta_{(\chi, w)}(E_\chi g_w, E_\chi g_w) \neq 0$. It is easy to see that they exactly correspond to the elements contained in the set stated in Theorem 2.5, if we identify $\text{Irr}(\mathcal{T})$ with C^n and set w to be the longest element of W_J , where W_J is the Young subgroup of \mathfrak{S}_n associated to J . \square

3. APPENDIX. REPRESENTATIONS OF NIL YOKONUMA-HECKE ALGEBRAS

In the appendix, we present two different approaches to classifying the simple modules of the nil Yokonuma-Hecke algebra ${}^0Y_{r,n}^f$ over an algebraically closed field \mathbb{K} of characteristic p such that p does not divide r .

${}^0Y_{r,n}^f$ is a \mathbb{K} -associative algebra generated by the elements T_1, \dots, T_{n-1} and t_1, \dots, t_n with relations:

$$\begin{aligned} t_i^r &= 1 & \text{for all } i = 1, \dots, n; \\ t_i t_j &= t_j t_i & \text{for all } i, j = 1, \dots, n; \\ T_i t_j &= t_{s_i(j)} T_i & \text{for all } i = 1, \dots, n-1 \text{ and } j = 1, \dots, n; \\ T_i T_j &= T_j T_i & \text{for all } i, j = 1, \dots, n-1 \text{ such that } |i-j| \geq 2; \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} & \text{for all } i = 1, \dots, n-2; \\ T_i^2 &= 0 & \text{for all } i = 1, \dots, n-1. \end{aligned} \quad (3.1)$$

Let $w \in \mathfrak{S}_n$ be with a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Then $T_w := T_{i_1} \cdots T_{i_r}$ is independent of the choice of the reduced expression of w . Let ${}^0H_n^f$ denote the subalgebra of ${}^0Y_{r,n}^f$ generated by T_1, \dots, T_{n-1} , which is exactly the nil Hecke algebra. Let $\mathbb{K}T$ denote the subalgebra generated by t_1, \dots, t_n . It is easy to see that the elements

$$\{t_1^{a_1} \cdots t_n^{a_n} T_w \mid 0 \leq a_1, \dots, a_n \leq r-1 \text{ and } w \in \mathfrak{S}_n\} \quad (3.2)$$

form a \mathbb{K} -basis of ${}^0Y_{r,n}^f$.

The following proposition gives a classification of simple modules of ${}^0Y_{r,n}^f$.

Proposition 3.1. *The Jacobson radical of ${}^0Y_{r,n}^f$ is the two-sided ideal generated by the elements T_w for all $w \neq 1$. Moreover, there are r^n non-isomorphic simple modules of ${}^0Y_{r,n}^f$, which are all of dimension 1.*

Proof. Since $J^{n(n-1)+1} = 0$ and ${}^0Y_{r,n}^f/J \simeq \mathbb{K}T$, we have $J = \text{rad}({}^0Y_{r,n}^f)$. Moreover, all the simple modules of ${}^0Y_{r,n}^f$ are of dimension 1, which are indexed by $\text{Irr}(T)$, where $\text{Irr}(T)$ denotes the set of characters of $\mathbb{K}T$. In fact, for each $\chi \in \text{Irr}(T)$, the associated simple module η_χ is defined by

$$\eta_\chi(t_j) = \chi(t_j) \quad \text{for } 1 \leq j \leq n \quad \text{and} \quad \eta_\chi(T_i) = 0 \quad \text{for } 1 \leq i \leq n-1.$$

Since $t_j E_\chi T_{w_0} = \chi(t_j) E_\chi T_{w_0}$ for $1 \leq j \leq n$ and $T_i E_\chi T_{w_0} = 0$ for $1 \leq i \leq n-1$. Thus, η_χ can also be realized as the minimal left ideal ${}^0Y_{r,n}^f E_\chi T_{w_0}$ in the regular representation. In fact, the $\mathbb{K}E_\chi T_{w_0}$, for $\chi \in \text{Irr}(T)$, give all the minimal non-zero two-sided ideals. \square

There is an involution ψ on ${}^0Y_{r,n}^f$ defined by $\phi(T_i) = T_{n-i}$ for $1 \leq i \leq n-1$ and $\phi(t_j) = t_{n+1-j}$ for $1 \leq j \leq n$. Let $\lambda : {}^0Y_{r,n}^f \rightarrow \mathbb{K}$ be a linear map by $\tau(t_1^{a_1} \cdots t_n^{a_n} T_w) = \delta_{w,w_0}$. Then we have the following result.

Proposition 3.2. *${}^0Y_{r,n}^f$ is a Frobenius algebra with a Frobenius form λ . Moreover, we have $\lambda(xy) = \lambda(\psi(y)x)$ for all $x, y \in {}^0Y_{r,n}^f$.*

We also have the following result.

Theorem 3.3. *${}^0Y_{r,n}^f$ is a standardly based algebra with a standard basis $\{E_\chi T_w \mid \chi \in \text{Irr}(T) \text{ and } w \in \mathfrak{S}_n\}$. Moreover, the simple modules of ${}^0Y_{r,n}^f$ over \mathbb{K} are exactly those which are given in Proposition 3.1.*

Proof. Since we have

$$T_i T_w = \begin{cases} T_{s_i w} & \text{if } \ell(s_i w) > \ell(w), \\ 0 & \text{if } \ell(s_i w) < \ell(w), \end{cases}$$

we get that

$$E_{\chi'} T_i \cdot E_\chi T_w = E_{\chi'} E_{s_i(\chi)} T_i T_w = \begin{cases} 0 & \text{if } s_i(\chi) \neq \chi', \\ E_{\chi'} T_{s_i w} & \text{if } s_i(\chi) = \chi' \text{ and } \ell(s_i w) > \ell(w), \\ 0 & \text{if } s_i(\chi) = \chi' \text{ and } \ell(s_i w) < \ell(w). \end{cases} \quad (3.3)$$

Set $\Delta := \{(\chi, w) \mid \chi \in \text{Irr}(T) \text{ and } w \in \mathfrak{S}_n\}$. We fix a total ordering on the elements of \mathfrak{S}_n such that $w_i > w_j$ whenever $\ell(w_i) > \ell(w_j)$ and extend it to Δ . Assume that $I(\chi, w)$ and $J(\chi, w)$ consist of only one element $E_\chi T_w$ for each $(\chi, w) \in \Delta$. By (3.3), we see that the basis $\{E_\chi T_w\}$ is a standard basis.

In order to classify the simple modules of ${}^0Y_{r,n}^f$ using [DR, Theorem 2.4.1], we must choose those (χ, w) such that $\beta_{(\chi, w)}(E_\chi g_w, E_\chi g_w) \neq 0$. Since $T_{w_1} T_{w_2} = T_{w_1 w_2}$ if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ and $T_{w_1} T_{w_2} = 0$ otherwise, it is easy to see that $\beta_{(\chi, w)}(E_\chi T_w, E_\chi T_w) \neq 0$ if and only if $w = 1$. \square

Acknowledgements. The author was partially supported by the National Natural Science Foundation of China (No. 11601273).

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